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A Class of Weak Chebyshev Spaces and Characterization of Best Approximations*

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Let $A \subset \mathbb{R}$, $F(A)$ denote the linear space of all real functions on A. A finite-dimensional subspace $U \subset F(A)$ is said to satisfy (WT_r)-property if every its restriction $U|_{A'}$, $A' \subset A$, is a weak Chebyshev space. It is shown that a direct extension of the characterization of the best approximations by spline-functions holds true for every WT_r-space. © 1995 Academic Press, Inc.

1. INTRODUCTION

Rice [7] and Schumaker [10] have independently presented the complete characterization of the best approximations of continuous functions by splines. They have shown that a necessary and sufficient condition for a polynomial spline s^* of degree m and with k fixed knots $a < x_1 < \cdots < x_n < b$ to be a best Chebyshev approximation on $[a, b]$ to a continuous function f is that $f - s^*$ possesses $m + p + 1$ points of alternation in some subinterval $[x_i, x_{i+p}]$, $0 \leqslant i < p \leqslant k + 1$ $(x_0 = a,\ x_{k+1} = b)$. If S denotes the linear space of all such splines, then $\dim S = m + k + 1$ and $\dim S|_{[x_i, x_{i+p}]} = m + p$, where $S|_{[x_i, x_{i+p}]} = \{s|_{[x_i, x_{i+p}]} : s \in S\}$ $(s|_{[x_i, x_{i+p}]}$ denotes the restriction of s to the interval $[x_i, x_{i+p}])$. It follows that a spline $s^* \in S$ is a best Chebyshev approximation to f if and only if there exists a subinterval $[\alpha, \beta] \subset [a, b]$ such that the maximal number of alternation points of $f - s^*$ in $[\alpha, \beta]$ exceeds the dimension of $S|_{[\alpha, \beta]}$.

This form of the characterization theorem looks rather general and the aim of the paper is to describe those spaces of real functions which admit such characterization of the best approximations.

We show that these are WT_r-spaces; i.e., those function spaces whose restrictions to all subsets of the domain of definition are WT-spaces. We also show that (WT_r)-property is intermediate between (WT)- and

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(WD)-properties and, therefore, our characterization theorem improves Micchelli's theorem [4] obtained for WD-spaces.

Note that our investigation has been inspired by the paper of Nürnberger *et al.* [5], where a characterization of the best approximations by generalized splines was established. The spaces of generalized splines, which were discussed there, are readily seen to have (WT_r)-property.

In Section 2 we give some definitions and state a general characterization theorem for the best approximations. This material is applied in the subsequent analysis. In Section 3 (WT_r)-property is introduced and its relationships to (WT)- and (WD)-property are investigated. In Section 4 we study some equivalent characterizations of WT_r-spaces in terms of its interpolation or alternation prperties. Finally, in Section 5, we present a characterization of the best approximations by WT_r-spaces. We also show that such a characterization does not hold when (WT_r)-property fails.

2. Characteristic Sets

Let $F(A)$ denote the linear space of all real functions on an arbitrary set A; if A is a compact Hausdorff space, $C(A)$ denotes the normed linear space of continuous funcions in $F(A)$ with the uniform norm $\|f\| = \max\{|f(t)| : t \in A\}$.

Suppose U is a finite-dimensional subspace of $F(A)$. For every subset A' of A, $U|_{A'}$ denotes the linear subspace of $F(A')$ consisting of the restrictions $u|_{A'}$ of all functions $u \in U$, i.e., $U|_{A'} = \{u|_{A'} \in F(A') : u \in U\}$, where $u|_{A'}(t) := u(t)$, $t \in A'$. A finite subset $A' \subset A$ is said to be an *interpolation set* or *I-set* relative to U if $\dim U|_{A'} = \operatorname{card} A'$. If $A' \subset A$ fails to be an I-set, we call it an NI-set. Every minimal NI-set is called a *characteristic set* or *C-set*.

It is easy to check that a finite subset $A' \subset A$ is a C-set relative to U if and only if we have

$$\begin{aligned} \dim U|_{A'} &= \operatorname{card} A' - 1, \\ \forall t \in A' \ \dim U|_{A' \setminus \{t\}} &= \operatorname{card} A' - 1. \end{aligned} \tag{1}$$

It has been shown by Dierieck [3] that A' satisfies (1) if and only if it is a minimal H-set relative to U or, equivalently, the support of a primitive extremal signature (see [1, 8] for definitions). This means that for every C-set $A' = \{t_1, ..., t_p\}$ there exists a sign pattern $\mathbf{e}^{A'} = \{e_1^{A'}, ..., e_p^{A'}\}$, $e_i^{A'} = +1$ or -1, $i = 1, ..., p$, called a *signature*, such that no function $u \in U$ satisfies

$$e_i^{A'} u(t_i) > 0, \qquad i = 1, ..., p. \tag{2}$$

Conversely, if a sign vector $\mathbf{e} = (e_1, ..., e_p)$, $|e_i| = 1$, differs from $\mathbf{e}^{A'}$ and $-\mathbf{e}^{A'}$, then there exists a function $u \in U$ satisfying

$$e_i u(t_i) > 0, \qquad i = 1, ..., p. \tag{3}$$

Hence the signature $\mathbf{e}^{A'}$ of C-set A' is determined uniquely up to a factor ± 1.

In [8] extremal signatures have been drawn on to characterize the best approximations by arbitrary finite-dimensional subspace $U \subset C(A)$. From this result we can immediately obtain the following characterization theorem.

THEOREM A. *Let A be a compact and $U \subset C(A)$ be a finite-dimensional subspace. A function $u^* \in U$ is a best Chebyshev approximation to $f \in C(A) \setminus U$ if and only if there exists a C-set $A' = \{t_1, ..., t_p\} \subset A$ and a signature $\mathbf{e}^{A'} = (e_1^{A'}, ..., e_p^{A'})$ relative to A' such that*

$$f(t_i) - u^*(t_i) = e_i^{A'} \|f - u^*\|, \qquad i = 1, ..., p. \tag{4}$$

Usually, general characterization theorems like Theorem A find applications in multivariate Chebyshev approximation (see, for example, [1, 3, 6, 13–15]). In this paper we use the concept of C-set and Theorem A to obtain a characterization of the best approximations by some finite-dimensional spaces of functions of one real variable.

3. WT_r-SPACES

In what follows, we assume that A is a subset of the real line $\mathbb{R}$. Let $\mathbf{a} = (a_1, ..., a_n)$ be a vector of real numbers. Following [11] we define the number of (strong) *sign changes* of $\mathbf{a}$ by $S^-(\mathbf{a})$, the number of sign changes in the sequence $a_1, ..., a_n$, where zeros are ignored. If $f \in F(A)$, we call $S_A^-(f) := \sup\{S^-[f(t_1), ..., f(t_n)] : t_1 < t_2 < \cdots < t_n \in A, n \in \mathbb{N}\}$ the number of (strong) *sign changes of f on A*.

A finite-dimensional subspace $u \subset F(A)$ is called a *weak Chebyshev space* or *WT-space* if $S_A^-(u) \leqslant \dim U - 1$ for every nonzero element $u \in U$. Every basis $u_1, ..., u_n$ of a WT-space U is called a *WT-system*.

We say that a sequence of functions $u_1, ..., u_n \in F(A)$ is a *weak Descartes system* or *WD-system* if every nontrivial linear combination $u = c_1 u_1 + \cdots + c_n u_n$ satisfies $S_A^-(u) \leqslant S^-(c_1, ..., c_n)$. The linear span of a WD-system is called a *WD-space*.

We also say that U satisfies *property* (WT) or (WD) if it is a WT-space or WD-space, respectively.

A subspace $U \subset F(A)$ will be called a WT_r-space if $U|_{A'}$ is a WT-space for every subset $A' \subset A$.

THEOREM 1. *(WT_r)-property is strictly intermediate between properties (WT) and (WD). That is, (WD) $\Rightarrow$ (WT_r) $\Rightarrow$ (WT) and (WT) $\not\Rightarrow$ (WT_r) $\not\Rightarrow$ (WD).*

Proof. It is trivial that every WT_r-space is a WT-space. One can also see that every WD-space is a WT_r-space. Indeed, let $u_1, ..., u_n$ be a WD-system on A and $U = \text{span}(u_1, ..., u_n)$. For each $A' \subset A$ we can find a subsystem $u_{i_1}, ..., u_{i_p}$ such that $u_{i_1}|_{A'}, ..., u_{i_p}|_{A'}$ is a basis for $U|_{A'}$. Then every function $u' \in U|_{A'}$ has a representation $u' = c_1 u_{i_1}|_{A'} + \cdots + c_p u_{i_p}|_{A'}$ and if we set $u = c_1 u_{i_1} + \cdots + c_n u_{i_p} \in U$, then $u'(t) = u(t)$, $t \in A'$. Hence $S^-_{A'}(u') = S^-_{A'}(u) \leqslant S^-_A(u) \leqslant S^-(c_1, ..., c_p) \leqslant p-1 = \dim U|_{A'} - 1$ and $U|_{A'}$ is seen to be a WT-space.

To show that $(WT) \not\Rightarrow (WT_r) \not\Rightarrow (WD)$ we give two examples.

EXAMPLE 1. A WT_r-space not satisfying (WD)-property: Let $A = [-2, 2]$, $u_1(t) := 0$ if $-2 \leqslant t < 0$, $u_1(t) := [1-(t-1)^2]^{1/2}$ if $0 \leqslant t \leqslant 2$, $u_2(t) := t$ for $t \in [-2, 2]$, and $U = \text{span}(u_1, u_2)$ [16, Example 4]. It is shown in [16] that U is a WT-space and does not contain two linearly independent nonnegative functions. Hence it fails to be a WD-space. However it is evident that $U|_{A'}$ is a WT-space if $A' \subset [-2, 2]$ and $\dim U|_{A'} = 2$. Suppose that $\dim U|_{A'} = 1$. If $A' \cap (0, 2] \neq \emptyset$, then $A' = \{t_0\}$ or $A' = \{0, t_0\}$ with $t_0 \in (0, 2]$ and every function $u \in U$ has no sign change in A'. If $A' \subset [-2, 0]$, then every $u \in U$ has constant sign in A'. Therefore, U is a WT_r-space.

EXAMPLE 2. A WT-space not satisfying (WT_r)-property: Let $A = [-2, 2]$, $u_1(t) = 1$ if $-1 \leqslant t \leqslant 1$, $u_1(t) = 0$ otherwise, $u_2(t) = t$ if $-1 \leqslant t \leqslant 1$, $u_2(t) = 0$ otherwise, $u_3(t) = 0$ if $-1 \leqslant t \leqslant 1$, $u_3(t) = |t| - 1$ if $-2 \leqslant t < -1$ or $1 < t \leqslant 2$, and $U = \text{span}(u_1, u_2, u_3)$. It is easy to verify that U is a WT-space on $[-2, 2]$. But setting $A' = \{-1.5, 0, 1.5\}$, we have $\dim U|_{A'} = 2$ and the function $u = u_3 - u_1$ has two sign changes in A'. Therefore $U|_{A'}$ fails to be a WT-space and U does not satisfy (WT_r)-property.

This completes the proof of Theorem 1.

A weak version of (WT_r)-property holds for every WT-space as it follows from the next theorem.

THEOREM B. *Let U be a WT-space on a set $A \subset \mathbb{R}$. Then $U|_{A'}$ is a WT-space if $A' = [\alpha, \beta] \cap A$, with $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$.*

The proof of Theorem B is equal to one given in [11] to prove Theorem 2.40 therein, which says, using our terminology, that every WT-space is a WT_r-space. This is not true, as Example 2 shows. But in the case when $A' = [\alpha, \beta] \cap A$, it is easy to check that the proof becomes correct. Furthermore, one can find a proof of Theorem B in [12, Thm. 1.4], where it is additionally assumed that $A = [a, b]$ and $U \subset C(A)$, but these suppositions are not applied.

4. Some Properties of WT_r-Spaces

Our purpose in this section is to characterize WT_r-spaces in terms of the properties of its I-sets and C-sets. The analysis will enable us to obtain our main result in Section 5 and, moreover, it seems to be of some independent interest.

Let U be a finite-dimensional subspace of $F(A)$ provided $A \subset \mathbb{R}$. If a finite subset $M \subset A$ is an I-set relative to U, then it is readily seen that $\forall \alpha, \beta \in \mathbb{R}$, $\alpha \leqslant \beta$,

$$\operatorname{card}(M \cap [\alpha, \beta]) \leqslant \dim U|_{A \cap [\alpha, \beta]} \tag{5}$$

(we set $[\alpha, \alpha] = \{\alpha\}$). We say that U satisfies (SW)-*property* if inequality (5) is a sufficient condition for M to be an I-set. It is significant that the space of polynomial splines satisfies property (SW). Really, this fact follows from the theorem of Schoenberg and Whitney [9] which characterizes I-sets relative to splines.

It is easily understood that if we know all I-sets relative to a space U, then we can describe all C-sets relative to it because they are minimal finite sets which fail to be I-sets. By these means we obtain the following equivalent definition of (SW)-property in terms of C-sets: a space U satisfies (SW)-property if and only if a necessary and sufficient condition for a finite set $M = \{t_1 < t_2 < \cdots < t_s\} \subset A$ to be a C-set is that

$$\dim U|_{A \cap [t_1, t_s]} = s - 1 \tag{6}$$

and

$$\dim U|_{A \cap [t_i, t_{i+s}]} = p + 1 \tag{7}$$

for all i, p with $1 \leqslant i \leqslant i + p \leqslant s$, $p < s - 1$.

A C-set $M = \{t_1 < t_2 < \cdots < t_s\}$ is said to be *alternating* if its signature $\mathbf{e}^M = (e_1^M, ..., e_s^M)$ satisfies the condition $e_i^M e_{i+1}^M = -1$, $i = 1, ..., s - 1$. We say that a finite-dimensional space U has *property* (AC) if every its C-set is alternating (it is evidently equivalent to the condition that $U|_M$ is a WT-space for every C-set M).

Theorem 2. *Let $A \subset \mathbb{R}$. Suppose U is a finite-dimensional subspace of $F(A)$. The following properties of U are equivalent to each other*:

(a) (WT_r);

(b) (SW) *and* (WT);

(c) (SW) *and* (AC).

Proof. First we show that $(WT_r) \Rightarrow (SW)$. Let U be a WT_r-space. We need to verify that every finite set $M \subset A$ satisfying (5) is an I-set relative to U or, equivalently, that every NI-set does not satisfy (5). Actually, let $M = \{t_1 < t_2 < \cdots < t_s\} \subset A$ be a NI-set. We show that there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\operatorname{card}(M \cap [\alpha, \beta]) > \dim U|_{A \cap [\alpha, \beta]}. \tag{8}$$

We see that it suffices to check (8) for the minimal NI-sets, i.e., C-sets. So let M be a C-set. Our statement will be established if we verify that

$$\dim U|_{A \cap [t_1, t_s]} \leqslant s - 1. \tag{9}$$

Suppose the contrary. Then

$$\dim U(M) = \dim U|_{A \cap [t_1, t_s]} - \dim U|_M \geqslant 1,$$

where

$$U(M) := \{u \in U|_{A \cap [t_1, t_s]} : u(t_i) = 0, i = 1, ..., s\}.$$

Because of this there exists $u_0 \in U$ such that $u_0(t_i) = 0$, $i = 1, ..., s$, and $u_0(t^*) > 0$ for some $t^* \in A \cap [t_1, t_s]$. Let $t_{i_0} < t^* < t_{i_0+1}$. Since U is a WT_r-space, $U|_M$ is a WT-space of dimension $s-1$. Therefore, there does not exist $u \in U$ satisfying $(-1)^i u(t_i) > 0$, $i = 1, ..., s$. This means that the signatures of the C-set M are $e_i^M = \pm(-1)^i$, $i = 1, ..., s$. Hence for every other sign vector **e** there exists $u \in U$ such that (3) holds. Specifically, there exists $u_1 \in U$ satisfying $(-1)^{i_0 - i} u_1(t_i) > 0$, $i = 1, ..., i_0$ and $(-1)^{i - i_0 - 1} u_1(t_i) > 0$, $i = i_0 + 1, ..., s$. It is readily seen that one can find a sufficiently large positive factor α such that the function $u = u_1 - \alpha u_0$ has s sign changes in the points $t_1, ..., t_{i_0}, t^*, t_{i_0+1}, ..., t_s$ and, therefore, u has s sign changes in the set $M \cup \{t^*\}$. But $\dim U|_{M \cup \{t^*\}} = s$ and this contradicts the assumption that $U|_{M \cup \{t^*\}}$ is a WT-space.

Thus, the implication $(WT_r) \Rightarrow (SW)$ has been proved. Since (WT) follows trivially from (WT_r), we have established that (a) $\Rightarrow$ (b).

To prove that (b) $\Rightarrow$ (c), it will be sufficient to show that (SW) and (WT) imply (AC). Let U satisfy (SW) and (WT) and let $M = \{t_1 < t_2 < \cdots < t_s\} \subset A$ be a C-set relative to U. If M fails to be alternating, then there exists $u \in U$ such that $(-1)^i u(t_i) > 0$, $i = 1, ..., s$. Hence u has $s-1$ sign changes in the set $A \cap [t_1, t_s]$. In view of (6) we have $\dim U|_{A \cap [t_1, t_s]} = s - 1$ and, therefore, $U|_{A \cap [t_1, t_s]}$ is not a WT-space. Then Theorem B shows that U also fails to have property (WT). Because of the contradiction we deduce that every C-set relative to U must be alternating. This means that U satisfies the condition (AC).

Finally, let us verify that (c) $\Rightarrow$ (a). It is easy to check that the properties (SW) and (AC) are inherited by restrictions: i.e., if U satisfies one of them, then $U|_{A'}$, $A' \subset A$, has the same property as well. Therefore it will be sufficient to prove that (SW) and (AC) imply (WT). Suppose that U has properties (SW) and (AC), but U fails to be a WT-space. Then one can find a finite set $M = \{t_1 < t_2 < \cdots < t_{n+1}\} \subset A$, $n = \dim U$, such that there exists $u \in U$ satisfying

$$(-1)^i u(t_i) > 0, \qquad i = 1, ..., n+1. \tag{10}$$

Let us consider the set $\mathscr{I}$ consisting of intervals $I_{ij} = [t_i, t_j]$, $1 \leqslant i \leqslant j \leqslant n+1$ (we set $I_{ii} = \{t_i\}$), such that $\dim U|_{A \cap [t_i, t_j]} \leqslant j - i$. We have $I_{1,n+1} \in \mathscr{I}$ and $I_{ii} \notin \mathscr{I}$, $i = 1, ..., n+1$. The set $\mathscr{I}$ is finite and naturally ordered. Hence, there exists a minimal interval $I_{pq} \in \mathscr{I}$, $p < q$, so that $\dim U|_{A \cap I_{pq}} = q - p$ and $\dim U|_{A \cap I_{ij}} \geqslant j - i + 1$ if $I_{ij} \subset I_{pq}$, $I_{ij} \neq I_{pq}$. This means that the subset $M' = \{t_p < \cdots < t_q\} \subset M$ satisfies (6) and (7) and, in view of (SW)-property, M' is a C-set relative to U. Because of (10) we see that M' fails to be an alternating C-set. Therefore, we have a contradiction with (AC)-property of U.

This completes the proof of Theorem 2.

Remark 1. Theorem 2 shows that (SW) and (WT) imply (AC), and (SW) and (AC) imply (WT). One can ask whether (AC) follows immediately from (WT) or (WT) follows immediately from (AC) without the additional assumption that (SW) holds. The answer to both is negative as the following two examples show.

EXAMPLE 3. A WT-space not satisfying (AC)-property: Let $A = [-2, 2]$, $u_1(t) = 1$ if $-1 \leqslant t \leqslant 1$, $u_1(t) = 0$ otherwise, $u_2(t) = t + 1$ if $-2 \leqslant t < -1$, $u_2(t) = 0$ if $-1 \leqslant t \leqslant 1$, $u_2(t) = t - 1$ if $1 < t \leqslant 2$, and $U = \mathrm{span}(u_1, u_2)$. It is readily seen that U satisfies (WT)-property. On the other hand the C-set $M = \{-1.5, 1.5\}$ is not alternating and, therefore, U fails to have (AC)-property.

EXAMPLE 4. An AC-space not satisfying (WT)-property: Let $A = [-2, 2]$, u_1 is the same as in Example 3, $u_2(t) = |t| - 1$ if $-2 \leqslant t < -1$ or $1 < t \leqslant 2$, $u_2(t) = 0$ otherwise and $U = \mathrm{span}(u_1, u_2)$. We have $\dim U = 2$ and $u_1 - u_2$ has two sign changes. Hence U fails to be a WT-space. However, every C-set relative to U consists of two points which both lie in either $[-1, 1]$ or $[-2, -1) \cup (1, 2]$. It is evident that such a C-set is alternating. Therefore, U satisfies (AC)-property.

Remark 2. One can easily see that in defining (WT_r)-property it is sufficient to demand that $U|_{A'}$ should be a WT-space for every finite subset

$A' \subset A$ with card $A' \leqslant \dim U + 1$. We can obtain a little more from the proof of Theorem 2. Indeed, in the proof of the implication $(WT_r) \Rightarrow (SW)$ we used only that (i) $U|_M$ is a WT-space for every C-set M and (ii) $U|_{M \cup \{t\}}$ is a WT-space if M is a C-set with card $M \leqslant \dim U$ and $t \in A \backslash M$ lies between min M and max M. It is clear that (i) $\Leftrightarrow (AC)$ and, therefore, (i) and (ii) imply (SW) and (AC). Thus, by Theorem 2, the conditions (i) and (ii) are sufficient for U to be a WT_r-space.

In the rest of this section it will be shown that the conditions (b) and (c) in Theorem 2 may be simplified if A and U satisfy some natural additional assumptions.

Following [11] we say that $t \in A$ is an *essential point* of A relative to U provided there exists $u \in U$ with $u(t) \neq 0$. A finite-dimensional space $U \subset F(A)$, where $A \subset \mathbb{R}$, will be called *regular* if from the conditions $u \in U$, $u(t_1) > 0$, $u(t_2) < 0$ with $t_1, t_2 \in A$, $t_1 < t_2$ it follows that there exists an essential point $t \in A$, $t_1 < t < t_2$, such that $u(t) = 0$. It is easily seen that U is regular in the case when A is a connected subset of $\mathbb{R}$ (i.e., A is an open, closed, or semiopen finite or infinite interval), all points of A are essential with respect to U, and U consists of continuous functions.

THEOREM 3. *Let U be regular. Then U is a WT_r-space if and only if it satisfies (SW)-property.*

Proof. In view of Theorem 2 we need to check that $(SW) \Rightarrow (WT)$.

Suppose the contrary: let U satisfy (SW)-property and fail to be a WT-space. Then there exists $u^* \in U$ and $t_i \in A$, $i = 1, ..., n+1$ ($n = \dim U$) such that $t_1 < \cdots < t_{n+1}$ and $(-1)^i u^*(t_i) > 0$, $i = 1, ..., n+1$. Because of the regularity of U, one can find essential points $\tau_1, ..., \tau_n \in A$ with $t_i < \tau_i < t_{i+1}$, $u^*(\tau_i) = 0$, $i = 1, ..., n$. It is readily seen that $M = \{\tau_i\}_{i=1}^n$ is a NI-set and in view of (SW)-property there exist intervals $[\alpha, \beta]$ such that

$$\operatorname{card}(M \cap [\alpha, \beta]) > \dim U|_{A \cap [\alpha, \beta]}.$$

Let us consider the set $\mathscr{I}$ consisting of intervals $I_{ij} = [\tau_i, \tau_j]$, $1 \leqslant i \leqslant j \leqslant n+1$ satisfying

$$\operatorname{card}(M \cap I_{ij}) > \dim U|_{A \cap I_{ij}}.$$

It is evident that $\mathscr{I}$ is non-empty. Since τ_i are essential points, $I_{ii} \notin \mathscr{I}$, $i = 1, ..., n$. Suppose $I_{pq} = [\tau_p, \tau_q]$ is a minimal interval belonging to $\mathscr{I}$. Then $p < q$. Let $M' = \{\tau_p, ..., \tau_{q-1}\}$ and $A' = A \cap [\tau_p, t_q]$. Then

$$\operatorname{card}(M' \cap [\alpha, \beta]) \leqslant \dim U|_{A' \cap [\alpha, \beta]}$$

for every $\alpha, \beta \in \mathbb{R}$. Considering that $U|_{A'}$ also satisfies (SW)-property, we deduce that M' is an I-set. On the other hand, $\dim U|_{A'} < \operatorname{card}(M \cap I_{pq}) = q-p+1$ and $\dim U|_{A'} \geqslant \operatorname{card} M' = q-p$, so that $\dim U|_{A'} = q-p$. We have $\dim U|_{A'} = \dim U|_{M'} + \dim U(M')|_{A'}$, where $U(M') := \{u \in U : \forall t \in M', u(t) = 0\}$. Since $u^* \in U(M')$ and $u^*(t_q) \neq 0$, $\dim U(M')|_{A'} \geqslant 1$. Therefore, $\dim U|_{M'} = \dim U|_{A'} - \dim U(M')|_{A'} \leqslant q-p-1 < \operatorname{card} M'$. This shows that M' fails to be an I-set. The contradiction proves Theorem 3.

5. Best Approximation by WT_r-Spaces and Concluding Remarks

Let $A \subset \mathbb{R}$ be a compact. We say that a function $\delta \in C(A)$ *possesses* s *points of alternation* in a subset $A' \subset A$ if there exist points $t_1, ..., t_s \in A'$ such that $t_1 < \cdots < t_s$ and $\delta(t_{i+1}) = -\delta(t_i) = \pm \|\delta\|$, $i = 1, ..., s-1$.

Now we state our main result.

Theorem 4. *Let $A \subset \mathbb{R}$ be a compact and $U \subset C(A)$ be a WT_r-space. A function $u^* \in U$ is a best Chebyshev approximation to $f \in C(A) \backslash U$ if and only if there exists an interval $[\alpha, \beta]$ such that the difference $\delta := f - u^*$ possesses $\dim U|_{A \cap [\alpha,\beta]} + 1$ points of alternation in $A \cap [\alpha, \beta]$.*

Proof. We consider the sufficiency first. If $\delta = f - u^*$ possesses $\dim U|_{A \cap [\alpha,\beta]} + 1$ points of alternation in $A \cap [\alpha, \beta]$ and $u_1 \in U$ is a better approximation than u^*, then $u_1 - u^*$ has $\dim U|_{A \cap [\alpha,\beta]}$ sign changes in $A \cap [\alpha, \beta]$ and $U|_{A \cap [\alpha,\beta]}$ fails to be a WT-space. This contradicts the assumption that U is a WT_r-space.

We consider the necessity next. Let u^* be a best approximation to f. In view of Theorem A there exists a C-set $M = \{t_1 < t_2 < \cdots < t_p\} \subset A$ and a signature $\mathbf{e}^M = (e_1^M, ..., e_p^M)$ such that $\delta(t_i) = e_i^M \|\delta\|$, $i = 1, ..., p$. It follows from Theorem 2 that U satisfies (SW) and (AC)-properties. Therefore,

$$\dim U|_{A \cap [t_1, t_p]} = p - 1$$

(cf. (6)) and $e_i^M = \pm(-1)^i$, $i = 1, ..., p$ so that δ possesses p points of alternation $t_1, ..., t_p$ in $A \cap [t_1, t_p]$. This concludes the proof.

Remark 3. It follows from the proof of Theorem 4 that the theorem holds true in the part of sufficiency if we consider any WT-space. Indeed, we can apply Theorem B instead of (WT_r)-property. By contrast, we cannot omit (WT_r)-property when proving the necessity. Actually, let U fail to be a WT_r-space. Then, by Theorem 2, it does not satisfy (SW)- or (AC)-property. Hence there exists a C-set $M = \{t_1 < t_2 < \cdots < t_s\}$ such that either (i) inequality (5) holds for all α, β or (ii) the signature $\mathbf{e}^M$ is not alternating. We can find a function $f \in C(A) \backslash U$ satisfying $f(t_i) = e_i^M$, $i = 1, ..., s$, and $|f(t)| < 1$ if $t \in A \backslash M$. It follows from Theorem A that $u^* = 0$ is a best approximation to f. In the case (i) every interval $A \cap [\alpha, \beta]$ contains at most card$(M \cap [\alpha, \beta]) \leqslant \dim U|_{A \cap [\alpha, \beta]}$ points of alternation of the function $\delta := f - u^* = f$. In the case (ii) δ possesses at most $s-1$ points of alternation in $A \cap [t_1, t_s]$ with $\dim U|_{A \cap [t_1, t_s]} \geqslant \dim U|_M = s-1$. If $A \cap [\alpha, \beta]$ does not contain M, then the number of alternation points of δ in $A \cap [\alpha, \beta]$ does not exceed card$(M \cap [\alpha, \beta]) = \dim U|_{M \cap [\alpha, \beta]} \leqslant \dim U|_{A \cap [\alpha, \beta]}$. As we see, in both cases our condition that δ possesses $\dim U|_{A \cap [\alpha, \beta]} + 1$ points of alternation in $A \cap [\alpha, \beta]$ fails.

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